Note

Law of Universal Gravitation 万有引力定律：

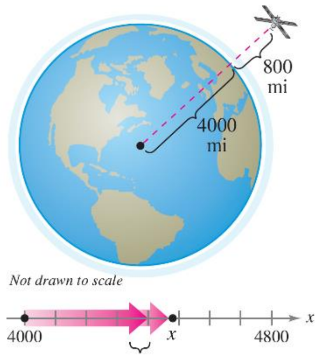
Newton's law of Gravitation 万有引力定律：

$$F_g = \frac{Gm_1m_2}{r^2} \qquad \mathrm{G} = 6.67 \times 10^{-11} N \cdot m^2/kg^2$$

在地球表面 $F_g = \frac{GM_E m}{R_e^2} = ma_g \Rightarrow a_g = g = \frac{GM_E}{R_e^2} = 9.8m/s^2$

Example: A space module weights 15 metric tons on the surface of Earth. How much work is done in propelling the module to a height of 800 miles above Earth, as shown in the figure.

- Use 4000 miles as the radius of Earth.
- Do not consider the effect of air resistance of the weight of the propellant.
- The weight of a body varies inversely as the Law of Universal Gravitation by $F_x = C/x^2$
  C is the constant of proportionality.



Solution On the surface of Earth:

$$F_x = \frac{C}{x^2} \Rightarrow C = F_x x^2 = 240{,}000{,}000$$

Calculate the work

$$W = \int_{x1}^{x2} F_x dx = \int_{4000}^{4800} \frac{240{,}000{,}000}{x^2} dx = \left(\frac{-240{,}000{,}000}{x}\right)_{4000}^{4800}$$
$$= 10{,}000 \ (mile \cdot ton) \approx 1.578 \times 10^{11} (Joule)$$

Exercise 14: Use 4000 miles as the radius of Earth. Neglecting air resistance and the weight of the propellant:

(a) determine the work done in propelling a five-ton satellite to a height of 100 miles above Earth.

(b) Write the work W of the propulsion system as a function of the height h of the satellite above Earth. Find the limit (if it exists) of W as h approaches infinity.

Exercise 15: In the "*Return of the Jedi*" Luke Skywalker asked an Ewok to fire a large drift bottle from a space station. The mass of the space station is 400,000 ton and the equivalent radius of it is 20m. The mass of the drift bottle is 10kg. ($G = 6.67 \times 10^{-11} N \cdot m^2/kg^2$)To make sure that the drift bottle can reach as far as possible to escape the gravitation of the space station (fly to infinity),

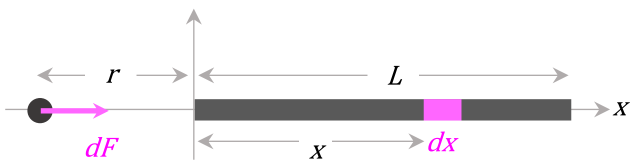
(a) what is the work must be done on it?

(b) Find the minimum initial velocity (v_0) of it.

Example: The left end of a thin uniform rod is placed at the origin and it lies along the x axis. The mass of rod is *M* and the length is *L*. A particle of mass m_a is placed at the coordinate (-*r*,*0*). Find the gravitational force due to the rod on the particle.

1) A small segment dx is at the distance x from the origin. The mass of this segment is dm.

$$dm = \frac{M}{L}dx$$



2) The gravitation force exerted on the particle is dF_g.

$$dF_g = \frac{Gm_a dm}{(x+r)^2} = \frac{Gm_a M}{(x+r)^2 L}dx$$

注意:此表达式中只有 1 个变量x,其他都是常量。

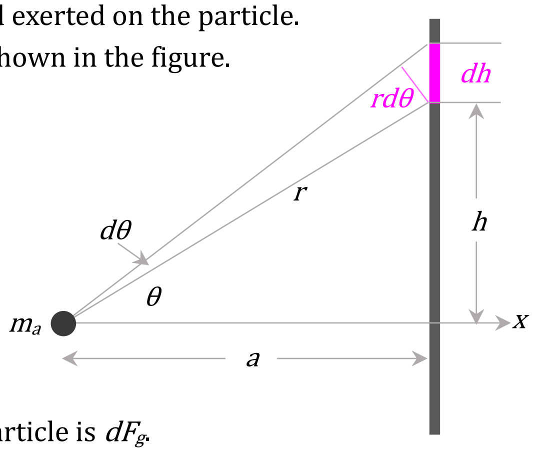
$$F_g = \int_0^L \frac{Gm_a M}{(x+r)^2 L}dx$$

$$u = x + r, du = dx, when\ x = 0\ u = r, when\ x = L\ \ u = r + L$$

$$F_g = \frac{Gm_a M}{L}\int_r^{r+L} u^{-2}du = \frac{Gm_a M}{L}\left(\frac{u^{-1}}{-1}\right)_r^{r+L} = \left(\frac{Gm_a M}{uL}\right)_{r+L}^{r}$$

$$= \frac{Gm_a M}{(r+L)L} - \frac{Gm_a M}{rL} = \frac{-Gm_a M}{r(L+r)}$$

Example: The distance between an infinitely long uniform rod and a particle is a (m). The mass per unit length of the rod is ρ (kg/m) and the mass of the particle is m_a (kg). Find the gravitational force of the rod exerted on the particle.



1) A small segment dh is at the height h shown in the figure.

   The mass of this segment is dm.

   $dm = \rho dh$

   The distance between this segment and the particle is r. $r^2 = a^2 + h^2$

   $$\cos\theta = \frac{rd\theta}{dh} \Rightarrow dh = \frac{rd\theta}{\cos\theta}$$

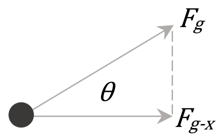
   $$\Rightarrow dm = \rho dh = \rho\frac{rd\theta}{\cos\theta}$$

2) The gravitation force exerted on the particle is dF_g.

   $$dF_g = \frac{Gm_a dm}{r^2} = \frac{Gm_a}{r^2}\left(\rho\frac{rd\theta}{\cos\theta}\right)$$

   dF_g在 h 方向上的分量dF_{g-h}是上下对称的，对 m_a 不起作用，

   只需要计算 x 方向上的分量dF_{g-x}



   $$dF_{g-x} = dF_g\cos\theta = \frac{Gm_a}{r^2}\left(\rho\frac{rd\theta}{\cos\theta}\right)\cos\theta = \frac{Gm_a\rho d\theta}{r}$$

   $$\cos\theta = \frac{a}{r} \Rightarrow r = \frac{a}{\cos\theta}$$

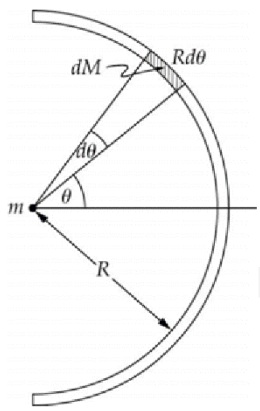
   $$dF_{g-x} = \frac{Gm_a\rho d\theta}{r} = \frac{Gm_a\rho d\theta}{a/\cos\theta} = \frac{Gm_a\rho\cos\theta\, d\theta}{a}$$

   注意:此表达式中只有 1 个变量θ,其他都是常量。

3) $F_{g-x} = \int_{-\pi/2}^{\pi/2} dF_{g-x} = \frac{Gm_a\rho}{a}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta$

   $$= \frac{Gm_a\rho}{a}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = \left(\frac{Gm_a\rho}{a}\sin\theta\right)_{-\pi/2}^{\pi/2} = \frac{2Gm_a\rho}{a}$$

Exercise 16: A thin uniform 20 kg rod with a length equal to 5.0 m is bent into a semicircle. What is the gravitational force exerted by the rod on a 0.10 kg point mass located at the center of curvature of the circular arc?



1) The unit mass of the rod is

2) The radius of the semicircle is

Exercise 14: Use 4000 miles as the radius of Earth. Neglecting air resistance and the weight of the propellant:

(a) determine the work done in propelling a five-ton satellite to a height of 100 miles above Earth.

On the surface of Earth: $F_x = \frac{C}{x^2} \Rightarrow C = F_x x^2 = 5 \cdot (4000)^2 = 80{,}000{,}000$

Calculate the work

$$W = \int_{x1}^{x2} F_x dx = \int_{4000}^{4100} \frac{-80{,}000{,}000}{x^2} dx = \left(\frac{-80{,}000{,}000}{x}\right)_{4000}^{4100} = 487.8\ (mile \cdot ton)$$

(b) Write the work W of the propulsion system as a function of the height h of the satellite above Earth. Find the limit (if it exists) of W as h approaches infinity.

$$W(h) = \int_{4000}^{4000+h} F_x dx = \int_{4000}^{4000+h} \frac{-80{,}000{,}000}{x} dx = 8 \times 10^7 \left(\frac{1}{4000} - \frac{1}{4000+h}\right)$$

$$\lim_{h\to\infty} W(h) = \lim_{h\to\infty} \left[8 \times 10^7 \left(\frac{1}{4000} - \frac{1}{4000+h}\right)\right] = \frac{8 \times 10^7}{4000} = 20000\ \ (mile \cdot ton)$$

Exercise 15: In the "*Return of the Jedi*" Luke Skywalker asked an Ewok to fire a large drift bottle from a space station. The mass of the space station is 400,000 ton and the equivalent radius of it is 20m. The mass of the drift bottle is 10kg. ($G = 6.67 \times 10^{-11} N \cdot m^2/kg^2$)To make sure that the drift bottle can reach as far as possible to escape the gravitation of the space station (fly to infinity),

(a) what is the work must be done on it?

Known: Mass of the space station: $M_s = 400{,}000\ ton = 4 \times 10^8 kg$

Mass of the bottle: $m_b = 10kg$

Radius of the space station: $r_0 = 20\ m$

Law of Universal Gravitation: $F(r) = \frac{GM_s m_b}{r^2}$

$$W(r) = \int_{r0}^{r} F_r dr = \int_{r0}^{r} \frac{GM_s m_b}{r^2} dr = \left(GM_s m_b \frac{-1}{r}\right)_{r0}^{r} = GM_s m_b \left(\frac{1}{r_0} - \frac{1}{r}\right)$$

$$\lim_{r\to\infty} W(r) = \lim_{r\to\infty} \left[GM_s m_b \left(\frac{1}{r_0} - \frac{1}{r}\right)\right] = \frac{GM_s m_b}{r_0} = \frac{(6.67 \times 10^{-11})(4 \times 10^8)(10)}{20}$$

$$= 0.01334\ (J)$$

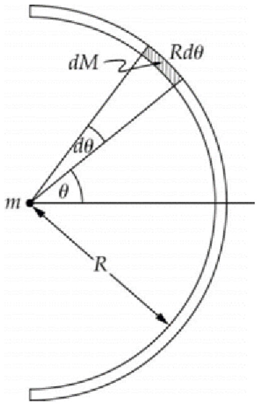
(b) Find the minimum initial velocity (v_0) of it.

Work= kinetic energy

$$\Rightarrow W = \frac{1}{2} m_b v_0^2 \Rightarrow v_0 = \sqrt{\frac{2W}{m_b}} = \sqrt{\frac{2 \times 0.01334}{10}} = 0.052\ \left(\frac{m}{s}\right)$$

Haha, Not a hard work.

Exercise 16: A thin uniform 20 kg rod with a length equal to 5.0 m is bent into a semicircle. What is the gravitational force exerted by the rod on a 0.10 kg point mass located at the center of curvature of the circular arc?



1) The unit mass of the rod is $\rho = \frac{20}{5} = 4\left(\frac{kg}{m}\right)$
2) The radius of the semicircle is $R = \frac{5}{\pi}\ (m)$
3) A small segment with length $dl = Rd\theta$ and mass $dm = \rho dl = \rho R d\theta$
4) The gravitational force $dF_g = \frac{Gm_a dm}{R^2}$, $m_a = 0.10kg$

   x component of dF_g is $dF_{g-x} = \frac{Gm_a dm}{R^2}\cos\theta$

$$F_{g-x} = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{Gm_a\rho R\cos\theta\, d\theta}{R^2} = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{Gm_a\rho\cos\theta\, d\theta}{R} = \left(\frac{Gm_a\rho\sin\theta}{R}\right)_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2Gm_a\rho}{R} = \frac{2(6.67\times10^{-11})(0.1)\times 4}{5/\pi} = 3.4\times10^{-11}\ (N)$$